AN EXOTIC SYMPLECTIC $\mathbf{R}^6$

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Abstract. An explicit example of an exotic symplectic $\mathbf{R}^6$ is given. Together with an earlier known example on $\mathbf{R}^4$, this yields an explicit exotic symplectic form on $\mathbf{R}^{2n}$ for all $n \geq 2$.

1. Introduction

Darboux' theorem states that there is a symplectic chart in the neighbourhood of any point in a symplectic manifold. This means that the symplectic form ω of any symplectic manifold locally looks like the canonical symplectic form ω_0 on $\mathbf{R}^{2n}$, that is, there are local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in which ω has the form

$$\omega = \sum_{a=1}^{n} dx_a \wedge dy_a.$$

A symplectic structure on $\mathbf{R}^{2n}$ is *exotic* if such a Darboux chart cannot be extended to a global symplectic embedding. It was of considerable interest when Gromov proved the existence of such a structure in [4] by first proving a nonvanishing theorem. More precisely, he proved the following.

Theorem 1.1. *For a compact embedded Lagrangian submanifold L in the standard symplectic structure $(\mathbf{R}^{2n}, \omega_0)$, the relative class $[\omega_0] \in H^2(\mathbf{R}^{2n}, L, \mathbf{R})$ never vanishes.*

He then gave an existence proof for a symplectic form ω on $\mathbf{R}^4$ with an exact Lagrangian torus (the class $[\omega] = 0 \in H^2(\mathbf{R}^{2n}, L, \mathbf{R})$), and hence an exotic symplectic structure. Later, Muller proved the existence of a symplectic structure on $\mathbf{R}^6$ with an embedded Lagrangian three-sphere [5], which, by Gromov's theorem, must be exotic. Similarly, Cho and Yoon have shown the existence of a symplectic $\mathbf{R}^4$ with a compact exact Lagrangian submanifold of genus two [2]. Unfortunately, the proofs of these theorems are non-constructive. It is therefore of interest to be able to write such a structure down explicitly. An example on $\mathbf{R}^4$ was given in [1]. In this note, we consider the case of $\mathbf{R}^6$.

2. The construction

An example of an exotic symplectic $\mathbf{R}^6$ will be constructed by giving the symplectic potential. Let $(x_1, x_2, x_3, y_1, y_2, y_3)$ be coordinates on $\mathbf{R}^6$. Consider the one forms

$$\psi_a := \frac{1}{2}\left(1 - \frac{7}{6}\left(x_a^2 + y_a^2\right) + \frac{1}{6}\left(x_a^2 + y_a^2\right)^2\right)(x_a\, dy_a - y_a\, dx_a) \quad a = 1, 2, 3$$

and set

$$\psi := \psi_1 + \psi_2 + \psi_3.$$

Observe that $\psi \equiv 0$ on the three torus T defined by

$$\left\{x_1^2 + y_1^2 = 1\right\} \cap \left\{x_2^2 + y_2^2 = 1\right\} \cap \left\{x_3^2 + y_3^2 = 1\right\}.$$

Define the function h by

$$h = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2\right),$$

and let $S = h^{-1}(3/2)$ be the sphere of radius $\sqrt{3}$ centered at the origin.

Claim 2.1. *The form $dh \wedge d\psi \wedge d\psi$ has rank five on the sphere S.*

Proof. Verification of the claim is summarized in the appendix. q.e.d.

Note that because the sphere S is a level set of h, this implies that the pullback of $d\psi \wedge d\psi$ to S has rank four everywhere on S. Define, in a neighbourhood of S, the one form χ by

$$\chi := *(dh \wedge d\psi \wedge d\psi),$$

where $*$ is the Hodge star operator in $\mathbf{R}^6$. Note that $\chi \neq 0$ in a neighbourhood of S. Set $s = \sqrt{2h} - \sqrt{3}$, so that $S = \{s = 0\}$. This allows the identification of the normal bundle $\mathrm{pr} : N \to S$ of the sphere S with a neighbourhood of the sphere. Finally, define a two form ω by

$$\omega := d(\mathrm{pr}^*(\psi|_S) + s\chi).$$

Theorem 2.2. *The two form ω is a symplectic form in a neighbourhood of S. Furthermore, the torus T is an exact Lagrangian submanifold.*

Proof. The form ω is closed because it is the exterior derivative of a one form, and it is nondegenerate because

$$\omega \wedge \omega \wedge \omega = 3\, ds \wedge \chi \wedge d\psi \wedge d\psi + O(s)$$

in a neighbourhood of S. Furthermore, since $T \subset S$ and $\psi|_T \equiv 0$, T is an exact Lagrangian submanifold.

q.e.d.

Corollary 2.3. *The two form ω defines an exotic symplectic structure on* $\mathbf{R}^6$.

Proof. A neighbourhood of S is identified with the normal bundle of S, and by removing a point of S not in the torus T, and its normal fibre, we have a manifold diffeomorphic to $\mathbf{R}^6$ with an exact Lagrangian submanifold. q.e.d.

As a final note, we point out that any direct product of the exotic structures on $\mathbf{R}^4$ and $\mathbf{R}^6$ described in [1] and here is exotic. Since any integer $n \geq 2$ can be written as $n = 2m + k$ with $m \geq 0$ and $k = 0$ or 3, we can bootstrap these exotic structures to an exotic structure on $\mathbf{R}^{2n}$ for any $n \geq 2$.

References

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3. appendix

The object of this appendix is to verify Claim 2.1. First of all, the formula for the one forms ψ_a was inspired by the overtwisted contact structures of Eliashberg [3]. The coefficient is based on the polynomial p, which satisfies

$$p(x) := 1 - 7x/6 + x^2/6, \quad p(0) = 1, \quad p(1) = 0, \quad p(3) = -1.$$

The actual expression for the five form $dh \wedge d\psi \wedge d\psi$ is far too complicated to reproduce here, but the verification of the rank condition in Claim 2.1 can be simplified by observing that the entire construction is invariant under the action of the torus group $\mathbf{T}^3$, which acts by rotation in each (x_a, y_a) plane. This implies that we may set all $y_a = 0$ in the formulae, reducing the problem to verifying that there is no simultaneous zero of the three polynomials

$$x_3 q\left(x_1^2, x_2^2\right), \quad x_1 q\left(x_2^2, x_3^2\right), \quad x_2 q\left(x_3^2, x_1^2\right),$$

when restricted to the sphere $x_1^2 + x_2^2 + x_3^2 = 3$, where q is defined by the formula

$$q(x, y) := 2 - \frac{14}{3}(x + y) + \frac{205}{18} xy + x^2 + y^2 - \frac{7}{3}(x^2 y - x y^2).$$

It is easily seen that there are no simultaneous zeros of the three polynomials above with $x_1, x_2,$ or $x_3 = 0$. The nonexistence of a simultaneous zero is therefore equivalent to showing that

$$Q(x, y, z) := q(x, y)^2 + q(y, z)^2 + q(z, x)^2$$

has a positive minimum on the simplex $\sigma = \{(x, y, z) : x + y + z = 3;\ x, y, z \geq 0\}$. It was found by means of *Mathematica* that Q is bounded below by 7×10^{-3} in the ball of radius 3 centered at the origin. Since σ is contained in this ball, this verifies the claim.

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